Langrange's equations of motion for conservative System :—

Consider a system of N particles. The transformation equations for the position vectors of the particles are-

$$r_i = r_i(q_1, q_2, \ldots q_k, \ldots q_n, t) \quad \text{——— (1)}$$

Where 't' is the time and q_k $(k=1,2,\ldots n)$ are the generalised coordinates.

Differentiating eqn (1) with respect to 't', then we get velocity of the ith particle, i.e

$$\frac{dr_i}{dt} = \frac{\partial r_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial r_i}{\partial q_2}\frac{dq_2}{dt} + \frac{\partial r_i}{\partial q_3}\frac{dq_3}{dt} + \ldots + \frac{\partial r_i}{\partial q_k}\frac{dq_k}{dt} + \ldots$$

$$+ \frac{\partial r_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial r_i}{\partial t}$$

$$\text{or, } v_i = \dot{r}_i = \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t} \quad \text{——— (2)}$$

where $\dot{q}_k$ are the generalised velocities.

The virtual displacement is

$$\delta r_i = \frac{\partial r_i}{\partial q_1}\delta q_1 + \frac{\partial r_i}{\partial q_2}\delta q_2 + \ldots + \frac{\partial r_i}{\partial q_k}\delta q_k + \ldots + \frac{\partial r_i}{\partial q_n}\delta q_n$$

$$\text{or, } \delta r_i = \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k}\delta q_k \quad \text{——— (3)}$$

since by definition the virtual displacements do not depends on time.

According to D'Alembert's principle,

$$\sum_{i=1}^{N}(F_i - \dot{P}_i)\cdot \delta r_i = 0 \quad \text{——— (4)}$$

Here,
$$\sum_{i=1}^{N} F_i \cdot \delta r_i = \sum_{i=1}^{N} F_i \cdot \sum_{K=1}^{N} \frac{\partial r_i}{\partial q_K}\delta q_K$$

$$= \sum_{K=1}^{n}\sum_{i=1}^{N}\left[F_i \cdot \frac{\partial r_i}{\partial q_K}\right]\delta q_K = \sum_{K=1}^{n} G_K\,\delta q_K \quad \text{—— (5)}$$

Where,
$$G_K = \sum_{i=1}^{N} F_i \cdot \frac{\partial r_i}{\partial q_K} = \sum_{i=1}^{N}\left[F_{xi}\frac{\partial x_i}{\partial q_K} + F_{yi}\frac{\partial y_i}{\partial q_K} + F_{zi}\frac{\partial z_i}{\partial q_K}\right] \quad (6)$$

are called the components of Generalised force associated with the generalised coordinates q_K.

Further,
$$\sum_{i=1}^{N} \dot{P}_i \cdot \delta r_i = \sum_{i=1}^{N} m_i \ddot{r}_i \cdot \sum_{K=1}^{n}\frac{\partial r_i}{\partial q_K}\delta q_K$$

$$= \sum_{K=1}^{n}\left[\sum_{i=1}^{N} m_i\ddot{r}_i \cdot \frac{\partial r_i}{\partial q_K}\right]\delta q_K \quad \text{—— (7)}$$

Now,
$$\sum_{i=1}^{N} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_K} = \sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i \dot{r}_i \cdot \frac{\partial r_i}{\partial q_K}\right) - m_i\dot{r}_i \cdot \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_K}\right)\right] \quad (8)$$

It is easy to prove that,

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_K}\right) = \frac{\partial}{\partial q_K}\left(\frac{dr_i}{dt}\right) = \frac{\partial v_i}{\partial q_K} \quad \text{—— (9)}$$

and,
$$\frac{\partial r_i}{\partial q_K} = \frac{\partial v_i}{\partial \dot{q}_K} \quad \text{—— (10)}$$

Therefore, eqn. (8) is

$$\sum_{i=1}^{N} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i v_i - \frac{\partial v_i}{\partial \dot{q}_k}\right) - m_i v_i \cdot \frac{\partial v_i}{\partial q_k}\right] \rightarrow (11)$$

Substituting in eqn. (7) is

$$\sum_{i=1}^{N} \dot{P}_i \cdot \delta r_i = \sum_{K=1}^{n} \sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i v_i \cdot \frac{\partial v_i}{\partial \dot{q}_k}\right) - m_i v_i \cdot \frac{\partial v_i}{\partial q_k}\right]\delta q_k$$

$$= \sum_{K=1}^{n}\left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_k}\left(\sum_{i=1}^{N}\frac{1}{2} m_i (v_i \cdot v_i)\right)\right\} - \frac{\partial}{\partial q_k}\left\{\sum_{i=1}^{N}\frac{1}{2} m_i (v_i \cdot v_i)\right\}\right]\delta q_k$$

$$= \sum_{K=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right]\delta q_k \quad \text{——} \ (12)$$

Where, $\sum_i \frac{1}{2} m_i (v_i \cdot v_i) = \sum_i \frac{1}{2} m_i v_i^2 = T$ is the Kinetic energy of the system.

Substituting for $\sum_i F_i \cdot \delta r_i$ from eqn (5) and $\sum_i \dot{P}_i \cdot \delta r_i$ from (12) in eqn (4), the D'Alembert's principle becomes,

$$\sum_{K=1}^{n}\left[\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right\} - G_k\right]\delta q_k = 0 \quad \text{——} \ (13)$$

As the constraints are holonomic, it means that any virtual displacement δq_k is independent of δq_js.

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Therefore, the coefficient in the square bracket for each δq_K must be zero, i.e

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_K}\right) - \frac{\partial T}{\partial q_K} - G_K = 0$$

or, $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_K}\right) - \frac{\partial T}{\partial q_K} = G_K \quad \text{———} \quad (14)$$

This represents the general form of Lagrange's equations

For a Conservative System :-

The force is derivable from a scalar potential V

$$F_i = \nabla_i V = -\hat{i}\frac{\partial V}{\partial x_i} - \hat{j}\frac{\partial V}{\partial y_i} - \hat{k}\frac{\partial V}{\partial z_i} \quad \text{———} \quad (15)$$

Hence, from eqn (6), the generalised force components are

$$G_K = -\sum_{i=1}^{N}\left[\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_K} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_K} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_K}\right] \quad \text{———} \quad (16)$$

clearly, the right hand side of the above equation is the partial derivative of $-V$ with respect to q_K, i.e

$$G_K = -\frac{\partial V}{\partial q_K}$$

Thus, eqn (14) assumes the form,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_K}\right) - \frac{\partial T}{\partial q_K} = -\frac{\partial V}{\partial q_K} \quad \text{———} \quad (17)$$

or, $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_K}\right) - \frac{\partial (T-V)}{\partial q_K} = 0 \quad \text{———} \quad (18)$$

Since, the scalar potential 'V' is the function of generalised Coordinates q_K only not depending on generalised velocities, eqn (18) becomes,

$$\frac{d}{dt}\left[\frac{\partial(T-V)}{\partial \dot{q}_k}\right] - \frac{\partial(T-V)}{\partial q_k} = 0 \quad \text{—— (19)}$$

Where $L = T-V$ is called the Lagrangian of the System.

Thus eqn. (19) takes the form,

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0} \quad \text{—— (20)}$$

where $k = 1, 2, \text{---} n$

These equations are known as Langrage's equation for Conservative system.